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A SIMPLE MAXIMUM PRINCIPLE

by

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Dies ist ein informeller Diskussionsbeitrag.

1. Introduction

PONTRJAGIN's Maximum Principle is becoming a standard tool of economic analysis. The purpose of this paper is to provide the student with a rather simple proof of a special version which is almost exclusively used in dealing with economic problems[1].

2. The Problem

Denote by $\mathbb{R}^k$ the Euklidean k-space. The vector $x(\tau) \in \mathbb{R}^n$ denotes the <u>state</u> of the considered economic system at time τ. We are interested in the development of the system during the time interval $[t_o, t_1] \subset \mathbb{R}$. The change $\dot{x}(\tau)$ of the state at any point of time $\tau \in [t_o, t_1]$ is subject to the constraint $\dot{x}(\tau) \in U$ where U is a closed subset of $\mathbb{R}^n$. Furthermore we require the time paths $\dot{x}(\cdot)$ to be piecewise continuous functions of τ[2]. The time path $x(\cdot)$ is determined by the <u>control</u> $\dot{x}(\cdot)$ together with the initial condition $x(t_o) = x_o$. The problem is to choose an admissible control such that the integral

$$(1) \qquad \int_{t_o}^{t_1} f(x(\tau), \dot{x}(\tau), \tau)\, d\tau$$

is maximized. We assume the function $f : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}$ to be continuously differentiable. More formally, (1) is to be maximized subject to

$$(2) \qquad x(t_o) = x_o \quad , \quad \dot{x}(\tau) \in U \text{ for all } \tau \in [t_o, t_1]$$

<u>Example.</u> Interpret x as the capital stock and $\dot{x}$ as net investment in the economy. If $\varphi(x, \dot{x})$ is the maximum consumption obtainable for given capital stock x and given investment $\dot{x}$, the integral over $f(x(\tau), \dot{x}(\tau), \tau) = e^{-\sigma \cdot \tau} \cdot \varphi(x(\tau), \dot{x}(\tau))$ gives the discounted sum of consumption for a rate of discount σ.

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1) The following has been presented in a course on the theory of investment held at the University of Regensburg in winter 1970/71.

2) Formally:

$$\dot{x}(\cdot) = \{(\tau, \dot{x}) \mid t_o \leq \tau \leq t_1 ,\ \dot{x} = \dot{x}(\tau)\}$$

3. The Derivation of the **Maximum** Principle

Now we intend to derive a <u>necessary</u> condition which must be satisfied if the integral (1) has assumed its maximum value. As will turn out in practical applications, this condition (**the Maximum Principle**) is often sufficient to determine the solution of the maximization problem if one knows from other reasons that there exists a solution at all.

Let $x(\cdot)$ be a solution of the maximization problem. Since we have

$$(3) \qquad x(t) = x_o + \int_{t_o}^{t} \dot{x}(\tau)\,d\tau$$

the time path $x(\cdot)$ remains unchanged if $\dot{x}(\cdot)$ is replaced by a control $\dot{x}'(\cdot)$ which differs from $\dot{x}(\cdot)$ at a finite number of points of time, and the integral (1) remains unchanged, too. This consideration leads us to the following conclusions:

1. Without loss of generality we can assume that **the right hand limit** $\lim_{\tau \downarrow t} \dot{x}(\tau)$ exists for all $\tau \in [t_o, t_1]$.

2. Any **admissible** control $\dot{x}'(\cdot)$ which differs from $\dot{x}(\cdot)$ only at a finite number of points of time is a solution of the maximization problem, too.

Now we start the main argument. We replace the control $\dot{x}(\cdot)$ by an **admissible** control $\dot{y}(\cdot)$ and use the fact that the value of the integral cannot be increased by this procedure. More formally: **We choose a number** $\alpha > 0$, a point of time $t \in [t_o, t_1 - \alpha]$ and an $u \in U$ and define the control $\dot{y}(\cdot)$ by

$$(4) \qquad \begin{aligned} \dot{y}(\tau) &= \dot{x}(\tau) \quad \text{for } \tau \in [t_o, t) \cup (t+\alpha, t_1] \\ \dot{y}(\tau) &= u \quad \text{for } \tau \in [t, t+\alpha] \end{aligned}$$

This control gives rise to a corresponding development of the state $y(\tau)$ of the system, since $y(t_o) = x_o$ from admissibility:

$$(5) \qquad \begin{aligned} y(\tau) &= x(\tau) \quad \text{for } \tau \in [t_o, t) \\ y(\tau) &= x(t) + (\tau - t)\cdot u \quad \text{for } \tau \in [t, t+\alpha) \\ y(\tau) &= x(\tau) + \int_{t}^{t+\alpha} \{u - \dot{x}(\sigma)\}\,d\sigma \quad \text{for } \tau \in [t+\alpha, t_1] \end{aligned}$$

Since $x(\cdot)$ is optimal and $y(\cdot)$ is admissible for any $u \varepsilon U$, we have

$$(6) \qquad \max_{u \varepsilon U} \left[\int_{t_o}^{t_1} f\{y(\tau),\dot{y}(\tau),\tau\}\, d\tau - \int_{t_o}^{t_1} f\{x(\tau),\dot{x}(\tau),\tau\}\, d\tau \right] \leq 0 \quad ^{1)}$$

Inserting (5) in (6) and dividing by α yields

$$(7) \qquad \max_{u \varepsilon U} \frac{1}{\alpha}\left[\int_{t_o}^{t} f\{x(\tau),\dot{x}(\tau),\tau\}\, d\tau + \int_{t}^{t+\alpha} f\{x(t)+(\tau-t)u,u,\tau\}\, d\tau + \right.$$

$$\left. + \int_{t+\alpha}^{t_1} f\{x(\tau)+\int_t^{t+\alpha}\{u-\dot{x}(\sigma)\}d\sigma,\dot{x}(\tau),\tau\}\, d\tau - \int_{t_o}^{t_1} f\{x(\tau),\dot{x}(\tau),\tau\}\, d\tau \right] \leq 0$$

This holds for all $t \varepsilon [t_o, t_1-\alpha]$ and reduces to

$$(8) \qquad \max_{u \varepsilon U} \left[\frac{1}{\alpha} \int_{t}^{t+\alpha} f\{x(t)+(\tau-t)u,u,\tau\}d\tau + \right.$$

$$\left. + \frac{1}{\alpha}\left\{ \int_{t+\alpha}^{t_1} f\{x(\tau)+\int_t^{t+\alpha} u-\dot{x}(\sigma))d\sigma,\dot{x}(\tau),\tau\}d\tau - \int_{t}^{t_1} f\{x(\tau),\dot{x}(\tau),\tau\}d\tau \right\}\right] \leq 0$$

for all $t \varepsilon [t_o, t_1-\alpha]$

For continuity reasons the inequality (8) remains true in the limit for $\alpha \downarrow 0$. In order to take this limit we apply L'HOSPITAL's Rule and find[2]

$$(9) \qquad \max_{u \varepsilon U} \lim_{\alpha \downarrow 0} \left[f\{x(t)+\alpha u,u,t+\alpha\} - f\{x(t+\alpha)+\int_t^{t+\alpha}\{u-\dot{x}(\sigma)\}d\sigma,\dot{x}(t+\alpha),t+\alpha\} + \right.$$

$$\left. + \int_{t+\alpha}^{t_1} f_x\{x(\tau)+\int_t^{t+\alpha}\{u-\dot{x}(\sigma)\}d\sigma,u,\tau\}d\tau\cdot\{u-\dot{x}(t+\alpha)\}\right] < 0$$

for all $t \varepsilon [t_o, t_1)$

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1) the maximum exists since U is closed and f is continuous.

2) see e.g. W. RUDIN, Principles of Mathematical Analysis, second edition, New York 1964, p. 94

where f_x denotes the vector of the partial derivatives of f with respect to its first n arguments. (The last term in the square brackets denotes the inner product of the vectors $\int f_x d\tau$ and $(u-\dot{x})$.)

(9) reduces to

$$(10) \qquad \max_{u \in U} \left[f\{x(t),u,t\} + u \cdot \int_t^{t_1} f_x\{x(\tau),\dot{x}(\tau),\tau\}\, d\tau \right] \leq$$

$$\leq \quad f\{x(t),\dot{x}(t),t\} + \dot{x}(t) \cdot \int_t^{t_1} f_x\{x(\tau),\dot{x}(\tau),\tau\}\, d\tau$$

$$\text{for all} \quad t \in [t_0,t_1)$$

There is only one additional step to be taken to obtain the Maximum Principle. We have to recall from the discussion of page 2 that any control which differs from $\dot{x}(\cdot)$ in at most a finite number of points of time, is an optimal control too. This implies that it is necessary for an optimal control that the inequality (10) is satisfied for almost all $t \in [t_0,t_1]$, i.e. for all t exept at most a finite number of points of time. Furthermore we note that the inequality sign in (10) can be replaced by strict equality since $\dot{x}(t) \in U$. Thus we conclude:

Theorem. Let f be a continuously differentiable function. A necessary condition that the integral (1) is maximized by a piecewise continuous control $\dot{x}(\cdot)$ satisfying $\dot{x}(\tau) \in U$ for all $\tau \in [t_0,t_1]$ is that

$$(11) \qquad \max_{u \in U} \left[f\{x,u,t\} + u \cdot \int_t^{t_1} f_x\{x,\dot{x},\tau\} d\tau \right] = f\{x,\dot{x},t\} + \dot{x} \cdot \int_t^{t_1} f_x\{x,\dot{x},\tau\} d\tau$$

holds for almost all points of time.

We remark that the proof and hence the theorem remains true for $t_1 = \infty$.

4. Remark

Although the above formulation of the Maximum Principle is most suitable with regard to the analysis of the given problem, it can be formulated equivalently in the following way.

Define

$$(12) \qquad \varphi(t) = \int_t^{t_1} f_x\{x,\dot{x},\tau\}d\tau$$

From (12) follows the equivalent definition of $\varphi(\cdot)$ by the system of differential equations

$$(13) \qquad \dot{\varphi} = - f_x(x,\dot{x},t) \qquad \text{for almost all } t \in [t_o,t_1]$$

together with the requirement of continuity and the boundary conditions

$$(14) \qquad \varphi(t_1) = 0$$

Now thě principle can be given the following formulation.

<u>Theorem.</u> Let f be a continuously differentiable function. A necessary condition that the integral (1) is maximized by a piecewise continuous control $\dot{x}(\cdot)$ satisfying $\dot{x}(\tau) \in U$ for all $\tau \in [t_o,t_1]$ is the existence of a continuous vector of functions $\varphi(t)$ satisfying (13) and (14) such that

$$(15) \qquad \max_{u \in U} \{ f(x,u,t) + u\cdot\varphi(t) \} = f(x,\dot{x},t) + \dot{x}\cdot\varphi(t)$$

is satisfied for almost all $t \in [t_o,t_1]$.[1)]

1) This is the formulation as given by Theorem 7 in L.S.PONTRJAGIN et al., Mathematische Theorie optimaler Prozesse, München 1967, and applied to our problem.