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The decomposition of a given time series into trend, seasonal component, and irregular component is formulated as a minimization problem. The trend is chosen such that it is as smooth as possible; the seasonal component is chosen such that it exhibits a seasonal pattern as stable as possible; and the trend and seasonal components are jointly chosen such that the given time series is explained as well as possible by these two components; that is, the irregular component is minimized.

KEY WORDS: Seasonal adjustment; Time series; Smoothing; Splines

1. INTRODUCTION

The problem of decomposing a given time series into trend, seasonal, and irregular components can be tackled in various ways (see Zellner 1978). Basically, there are two alternatives available. One can start from a statistical model that is supposed to have generated the time series and proceed to estimate the coefficients of the model. Alternatively, one can proceed in a purely descriptive fashion, smoothing out irregularities and distilling regular patterns through some averaging process.

The second, or descriptive, alternative will be pursued here. My preference for this approach rests on the observation that seasonal adjustment often serves the purpose of producing adjusted time series, which are then to be explained in terms of economic theory. If the adjustment method presupposes a statistical model, the economic model has to harmonize with the statistical model and this involves severe a priori restrictions.

The descriptive approach, on the other hand, ought to be transparent in the sense that the relationship between the original time series and the adjusted one can be easily understood. Otherwise, it is very difficult to assess the proper significance of the regularities detected in the adjusted time series. The aim of this article is to develop a transparent adjustment method.

There is a view—taken by De Vos and Wallis (oral communication)—stressing a duality rather than a dichotomy between the descriptive and the model-based approaches, a duality in the sense that for any particular adjustment method one can either invent a model leading to it or, alternatively, devise descriptive criteria implying this method. Still, transparency might very probably be on the side of the descriptive approach.

The second section of this article introduces the *seasonal adjustment principle*, as I call it. In the third section a seasonal adjustment method is derived by this principle, thus illustrating its meaning. Some remarks and comments follow in the fourth section.

2. THE SEASONAL ADJUSTMENT PRINCIPLE

The problem is to decompose a given time series $\{x_t\}_{t=1,2,\ldots,T}$ into a trend $\{y_t\}_{t=1,2,\ldots,T}$, a seasonal component $\{z_t\}_{t=1,2,\ldots,T}$, and an irregular component $\{u_t\}_{t=1,2,\ldots,T}$ such that

$$x_t = y_t + z_t + u_t \quad \text{for all } t = 1, 2, \ldots, T, \qquad (1)$$

or for short,

$$x = y + z + u, \qquad (2)$$

where x, y, z, and u are $T \times 1$ vectors with consecutively ordered components (i.e., $x' = (x_1, x_2, \ldots, x_T)$, etc.).

As for the shape of the trend and seasonal components, there are certain intuitive requirements implicit in the notions of trend and seasonal components.

1. The sum of trend y and seasonal component z should approximate the time series x as well as possible; that is, the irregular component $u = x - y - z$ is to be kept as small as feasible.

Let the function $h : \mathbb{R}^T \to \mathbb{R}$ be a measure of the size of u (e.g., $h(u) = u'u$, the sum of squares that we will actually employ later on; but it is not necessary to choose a norm or even a positive definite function). This first requirement translates into keeping $h(x - y - z)$ as small as possible.

2. The trend y is required to be a smooth curve depicting the long-run shape of x.

Let the function $f : \mathbb{R}^T \to \mathbb{R}$ be an (arbitrary) measure of curvature. The requirement of a smooth trend amounts to keeping $f(y)$ as small as possible.

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June 1981, Volume 76, Number 374
Theory and Methods Section

3. The seasonal component is required to exhibit a seasonal pattern that is as stable as possible. In addition, the seasonal components have to average to zero over the length of one season.

Introduce, again, a function $g : \mathbb{R}^T \to \mathbb{R}$ as a measure of the instability of the seasonal pattern and the deviation from an average of zero. Then the requirement of a regular seasonal component means that $g(z)$ is to be kept as small as possible.

The seasonal adjustment principle combines these three requirements. Given

1. A measure of curvature of the trend $f : \mathbb{R}^T \to \mathbb{R}$,
2. A measure of instability of the seasonal component $g : \mathbb{R}^T \to \mathbb{R}$,
3. A measure of the size of the irregular component $h : \mathbb{R}^T \to \mathbb{R}$,

the trend $y = y(x)$ and the seasonal component $z = z(x)$ are given by the solution of the following minimization problem:

$$\text{minimize } V(y, z \mid x) = f(y) + g(z) + h(x - y - z) \text{ with respect to } y \text{ and } z. \tag{3}$$

Thus the seasonal adjustment principle is a formalization of the intuitive requirement that the time series is to be decomposed into a smooth component (called trend) and a regular component (called seasonal component) such that the sum of these components explains the original time series as well as possible.

3. A SEASONAL ADJUSTMENT METHOD

In this section the seasonal adjustment principle just explained will be used to derive a specific seasonal adjustment method. This will be done by choosing particular functions f, g, and h.

The measure of curvature is taken as

$$f(y) := \alpha \cdot \sum_{t=3}^{T} \{(y_t - y_{t-1}) - (y_{t-1} - y_{t-2})\}^2, \quad \alpha > 0, \tag{4}$$

where the term $\{\ldots\}$ denotes the change of direction of the trend. These changes are squared and summed.

The measure of instability is taken as

$$g(z) := \beta \cdot \sum_{t=s+1}^{T} \{z_t - z_{t-s}\}^2 + \gamma \cdot \sum_{t=s}^{T} \left\{ \sum_{\tau=0}^{s-1} z_{t-\tau} \right\}^2, \quad \beta \geqq 0, \gamma > 0, \tag{5}$$

where s is a natural number larger than one and smaller than T, which denotes the length of the season (e.g., $s = 12$ in the case of monthly data). The first term in (5) measures the stability of the seasonal component over time. The second term is a measure of the deviation of the seasonal pattern from an average of zero.

The measure of the size of the irregular component is the sum of squares

$$h(x - y - z) := \sum_{t=1}^{T} \{x_t - y_t - z_t\}^2. \tag{6}$$

Once these functions have been specified, the seasonal adjustment principle can be applied and the following result is obtained.

Theorem: Given the specifications (4), (5), and (6), the minimization problem (3) has a unique solution $y(x)$, $z(x)$. Thus the seasonal adjustment principle yields in this case a unique decomposition of the given time series x into trend y, seasonal component z, and irregular component $u = x - y - z$.

Trend y and seasonal component z are uniquely defined by the equation

$$H \cdot \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x \end{pmatrix} \tag{7}$$

where

$$H := \begin{pmatrix} \alpha P'P + I & I \\ I & \beta Q'Q + \gamma R'R + I \end{pmatrix} \tag{8}$$

of order $(2T) \times (2T)$, where I denotes the $T \times T$ identity matrix, and

$$P := \begin{pmatrix} 1, & -2, & 1, & 0, & \ldots & & 0 \\ 0, & 1, & -2, & 1, & 0, & \ldots & \\ \vdots & & \ddots & & \ddots & \ddots & 0 \\ 0, & \ldots & & 0, & 1, & -2, & 1 \end{pmatrix} \tag{9}$$

of order $(T - 2) \times T$,

$$Q := \begin{array}{c} \text{column } s+1 \\ \downarrow \\ \begin{pmatrix} 1, & 0, & \ldots & 0, & -1, & 0, & \ldots & 0 \\ 0, & 1, & 0, & \ldots & 0, & -1, & 0, & \\ & \ddots & \ddots & & & \ddots & \ddots & 0 \\ 0, & \ldots & 0, & 1, & 0, & \ldots & 0, & -1 \end{pmatrix} \end{array}$$

of order $(T - s) \times T$, and

$$R := \begin{array}{c} \text{column } s \\ \downarrow \\ \begin{pmatrix} 1, & 1, & \ldots & 1, & 0, & \ldots & 0 \\ 0, & 1, & 1, & \ldots & 1, & 0, & \\ & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0, & \ldots & 0, & 1, & 1, & \ldots & 1 \end{pmatrix} \end{array} \tag{11}$$

of order $(T - s + 1) \times T$.

Proof: Expression (3) can be written as

$$V = \alpha \cdot y'P'Py + \beta \cdot z'Q'Qz + \gamma \cdot z'R'Rz + (x - y - z)'(x - y - z).$$

Upon differentiation one gets

$$\frac{1}{2}\frac{\partial V}{\partial\begin{pmatrix} y \\ z \end{pmatrix}} = H\cdot\begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} x \\ x \end{pmatrix} \tag{13}$$

$$\frac{1}{2}\frac{\partial^2 V}{\partial\begin{pmatrix} y \\ z \end{pmatrix}^2} = H \tag{14}$$

H is nonnegative definite because

$$H = G'G \tag{15}$$

where

$$G' := \begin{pmatrix} \sqrt{\alpha}P', & \sqrt{1/2}\cdot I, & 0, & \sqrt{1/2}\cdot I \\ 0, & \sqrt{1/2}\cdot I, & \sqrt{\beta}Q', \sqrt{\gamma}R', & \sqrt{1/2}\cdot I \end{pmatrix} \tag{16}$$

with zero-matrices 0 of appropriate order. Putting the first derivatives (13) to zero gives the necessary condition (7) for a minimum, which is also sufficient since H is nonnegative definite. If H is nonsingular, V is a strictly convex function with a unique minimum. The nonsingularity of H remains to be proved.

Since $H = G'G$, it follows that

$$\begin{aligned} r(H) = r(G') &\geqq r\begin{pmatrix} P', & 0, & I \\ 0, & R', & I \end{pmatrix} \\ &\geqq r\begin{pmatrix} P', & -R', & 0 \\ 0, & R', & I \end{pmatrix} \\ &= r\begin{pmatrix} P', & R', & 0 \\ & 0, & I \end{pmatrix} \\ &= r\begin{pmatrix} R \\ P \end{pmatrix} + T \geqq r(M) + T \end{aligned} \tag{17}$$

where M is a submatrix of $\begin{pmatrix} R \\ P \end{pmatrix}$

$$M = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$\begin{array}{c} \qquad\qquad\qquad\qquad\text{column } s \\ \qquad\qquad\qquad\qquad\downarrow \end{array}$$

$$= \left(\begin{array}{cc|ccccccc} 1, & 1, & & & 0, & \cdots & & & 0 \\ 0, & 1, & & & 1, & 0, & \cdots & & 0 \\ \hline 1, & -2, & 1, & 0, & \cdots & & & & 0 \\ 0, & 1, & -2, & 1, & 0, & & & & \\ & 0, & 1, & -2, & 1, & 0, & & & \\ & & 0, & \ddots & \ddots & \ddots & \ddots & & \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0, & 0, & 0, & & \cdots & 0, & 1, & -2, & 1 \end{array}\right) \tag{18}$$

It will be proved that M is nonsingular. Because M is of order $T \times T$, this implies $r(H) = 2T$, that is, nonsingularity of H. Since $\det D = 1$, $\det M = \det D\cdot\det(A - BD^{-1}C)$, and

$$D^{-1} = \begin{pmatrix} 1, & 0, & \cdots & & 0 \\ 2, & 1, & 0, & & \\ 3, & 2, & 1, & 0, & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ T-2, & \cdots & 3, & 2, & 1 \end{pmatrix} \tag{19}$$

one finds

$$\det M = \det(A - BD^{-1}C) = s^2, \tag{20}$$

which is positive.

The actual solutions to the normal equations (7) can be obtained by means of a computer program that has been developed by Nagel (1980). The program (available on request) yields a direct (i.e., noniterative) solution and is very fast and accurate.

4. THE CHOICE OF THE PARAMETERS

Two examples of how the method works are given here. There remains much freedom of course concerning the choice of weights α, β, and γ. Although this choice can be viewed as an ad hoc procedure, its implications are quite obvious: Increasing α will smooth the trend, and increasing β and γ will increase the stability of the seasonal pattern. Of course, this can be achieved only at the cost of increasing the size of the irregular component.

Alternatively, the choice of the parameters can be viewed not as an ad hoc procedure but, again, as an optimization problem: The parameters could be chosen, for example, such that the irregular component exhibits only a weak seasonal pattern. A more interesting principle is to minimize the ex post prediction errors during the last few years by a suitable choice of the weights. By ex post prediction error I mean the following: denote by x^t the vector of observations up to period $t < T$, and carry through the seasonal adjustment of the time series x^t for a given choice of weights. This will yield a seasonal component $z_t(\alpha, \beta, \gamma, t)$. Using the full time series leads to a seasonal component $z_t(\alpha, \beta, \gamma, T)$. The difference between these two seasonal components is the ex post prediction error. A similar calculation can be performed with respect to the trend component, and one can try to minimize the relative size of these ex post prediction errors (Nagel 1980, p. 15).

It is quite clear that the appropriate choice of parameters will depend on the shape of the time series. If the seasonal pattern is rigid over the whole period of observation, a very high α and γ would be appropriate. But this is not the case in the given example where the original time series exhibits a significantly varying seasonal pattern and therefore requires lower values of α and γ. Since

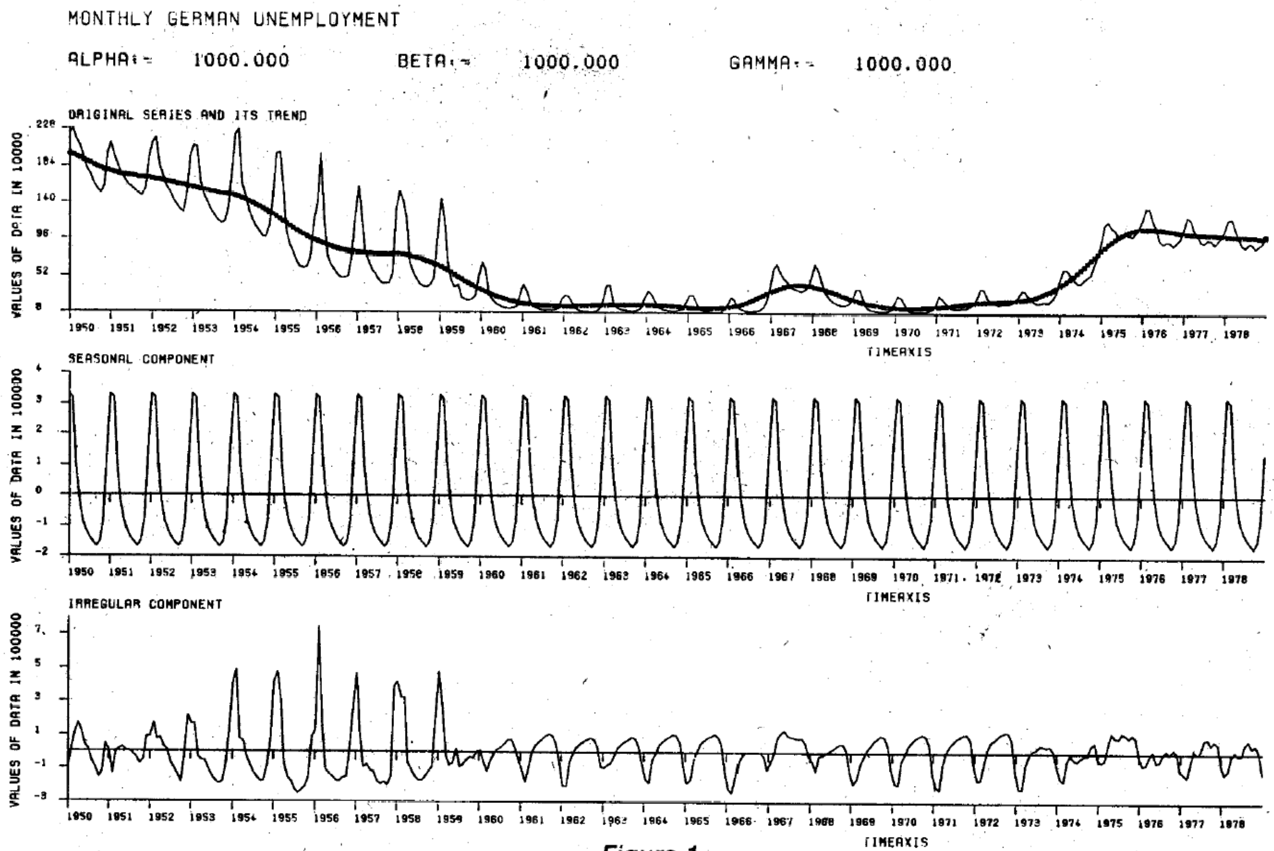

Figure 1

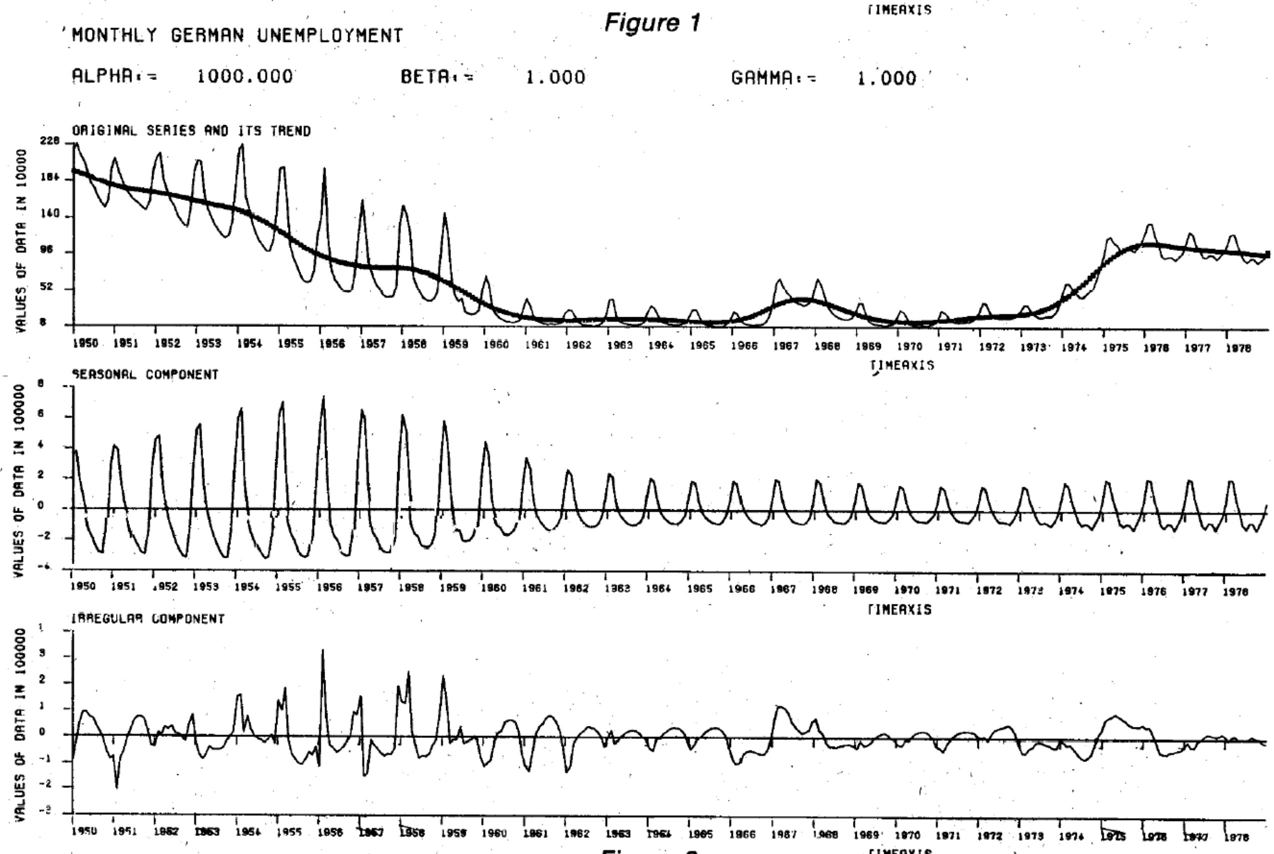

Figure 2

this is so, I consider the freedom to choose the parameters an advantage.

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