A Neoclassical Theory of Wealth Distribution

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1. Introduction

In his pioneering article on economic growth, Solow remarked: 'All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A "crucial" assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic'. [1, p. 58.] The first aim of this paper is to show that in the model of wealth distribution as proposed by Stiglitz [2], the assumption of a linear savings function is a crucial one in the above sense. If one replaces this assumption by the assumption of a convex savings function, i.e. an increasing marginal propensity to save, one arrives at qualitatively different conclusions. Since the analysis is considerably simplified by choosing the more sophisticated approach of Duesenberry's Relative Income Hypothesis (which allows furthermore for a simple introduction of Harrod-neutral technical progress into the model), this will be done in the present paper.

The second aim of this paper is to present a rather rigorous and complete analysis of the proposed model. Thus it is hoped to improve upon Stiglitz' rather heuristic discussion of his model. The third aim is to establish the proposition that the system produces stable two-class equilibria even if all individuals have identical behavioural equations (namely, identical savings functions). Therefore, our theory explains Pasinetti-type class savings behaviour endogeneously. Theorem 4 and Corollary 2 could be summarized by saying that the equilibrium with the highest capital intensity is stable provided that society is divided into many sufficiently small groups. As a consequence, if a two-class equilibrium is associated with a higher capital intensity than all other equilibria and the groups are sufficiently small, it is stable, and otherwise identical individuals own different amounts of wealth in this stable equilibrium[1]). (The last section discusses a similar property for nonstationary solutions.)

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[1]) It can be proved by the same methods that this result carries over to the case of a convex savings function as discussed by Stiglitz [2] in the section "Nonlinear Savings Functions". (Stiglitz' discussion is incorrect, consequently.)

2. The Model

2.1. Production and Factor Remuneration

As far as production is concerned, we follow Stiglitz in applying the standard neoclassical one-sector framework. Per capita production y is a twice continuously differentiable function $y\colon \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of capital intensity k satisfying the usual conditions[2]):

(1) $$y = y(k),\ y' > 0,\ y'' < 0,\ y(0) = 0,\ y'(0) = \infty,\ y'(\infty) = 0.$$

Our theory of factor remuneration will be a slightly generalized version of the marginal productivity doctrine[3]). We assume the wage rate w to be a strictly increasing continuously differentiable function $w\colon \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of capital intensity k, but not greater than the marginal product of labour. Thus, some aspects of monopolistic pricing can be taken into account.

(2) $$w = w(k),\ w' > 0,\ 0 < w \leq y - k \cdot y' \text{ if } k > 0,\ w(0) = 0.$$

Consequently the rate of interest r is a decreasing function of k and not less than the marginal productivity of capital. This follows from the book-keeping identity

(3) $$w + r \cdot k = y$$

and (2), i.e.

(4) $$\begin{aligned} r(k) &:= \frac{1}{k}\{y(k) - w(k)\} \geq y' \\ r' &= \frac{1}{k}\{(y' - r) - w'\} < 0. \end{aligned}$$

2.2. Wealth-Income Groups

Total population is divided into n groups. a_i denotes the fraction of total population belonging to the i-th group $(i = 1, 2, \ldots, n)$. Clearly, we have

(5) $$a_i \gneqq 0 \text{ for all } i,\ \Sigma a_i = 1.$$

All members of any one group have the same wealth holdings. Let c_i denote per capita wealth in group i. It is assumed that these wealth holdings are in terms of claims to the total capital stock in the economy. Therefore, capital intensity k is identical to average per capita wealth holdings:

(6) $$\Sigma a_i c_i = k$$

Furthermore it is assumed that all individuals receive the same labour income w. The income y_i of one individual of group i is therefore:

[2]) $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ is the set of nonnegative real numbers. As a matter of convenience we write f (∞) instead of $\lim_{k \to \infty} f(k)$, etc.

[3]) Cf. Stiglitz [2, p. 383 n. 4]

(7) $$y_i = w + r \cdot c_i = y + (c_i - k) \cdot r.$$

From (3), (5), (6) and (7) we verify that y is average income

(8) $$\Sigma a_i y_i = y.$$

2.3. Savings Behaviour

The main difference to Stiglitz' analysis is that his assumption of a constant marginal propensity to save is replaced by the assumption of an increasing marginal saving propensity. As will be seen later, this substitution produces some results different from those suggested by Stiglitz.

Consider a member of group i receiving income y_i. Let s^i denote the corresponding amount of savings. We assume s^i to be an increasing function of y_i with an increasing marginal propensity to save which approaches unity with increasing income. This postulate of an increasing marginal propensity to save is the usual textbook assumption which can be motivated by the conjecture that increasing income causes the "direct" wants of consumption to recede in importance as compared with the "indirect" wants of securing economic independence and a high social status by the accumulation of wealth.

According to the Relative Income Hypothesis, s^i is supposed to be influenced not only by individual income y_i, but as well by the general standard of living as represented by average income y, cf. [4, ch. 3 and 4] and [5, ch. 6]. If both y_i and y increase by a common factor, s^i is increased in the same proportion, i.e. the individuals' average saving propensity s^i/y_i is entirely determined by the relative income position of the individual (as represented by y_i/y) and independent of the absolute level of income.

To keep things simple, we define the savings function only for non-negative arguments and assume the corresponding amounts of saving to be nonnegative[4]).

Let all individuals behave alike, i.e. all having the same savings function s. Thus s^i is seen to be a twice continuously differentiable linear homogenous function $s: \mathbb{R}^2_+ \to \mathbb{R}_+$ of y_i and y with the following properties: (Subscripts indicate partial derivatives.)

(9) $$s^i = s(y_i,y) \geq 0, 0 \leq s_1 < 1, s_1(\infty,y) = 1, s_{11} > 0, s(0,y) = 0.$$

Linear homogeneity and continuity imply together with (9)

(10) $$s_2(y_i,y) = s\{y_i/y,1\} - s_1\{y_i/y,1\} \cdot y_i/y < 0, s(y_i,0) = y_i$$

i.e., an increase in average income y associated with constant individual income y_i reduces savings whereas a decrease causes the average saving propensity to approach unity.

[4]) Indeed, if the savings function were defined for negative arguments as well, a nondecreasing marginal saving propensity would imply negative consumption for sufficiently negative individual income. The nonnegativity of savings (for positive income) is postulated with the intention to avoid violations of the nonnegativity

2.4. The Accumulation Equations

In every group, population grows with the common rate $\eta > 0$. Therefore, the a_i's remain constant through time. Finally assume that children inherit equal parts of their parents' wealth. Now, the change of per capita wealth through time is given by

$$\dot{c}_i = s^i - \eta \cdot c_i \tag{11}$$

i.e., per capita wealth is increased by savings and decreased by $\eta \cdot c_i$ as a consequence of population growth.

By means of (1), (4), (6), (7), and (9), (11) can be rewritten as

$$\dot{c}_i = s\{y(k) + (c_i - k)r(k), y(k)\} - \eta\, c_i, k = \Sigma a_j c_j, i,j = 1,2,\ldots,n. \tag{12}$$

By means of the auxiliary function ψ defined as

$$\psi(c,k) := s\{y(k) + (c - k)r(k), y(k)\} - \eta c \tag{13}$$

the system (12) can be represented in a more compact way as

$$\dot{c}_i = \psi\{c_i, \Sigma a_j c_j\} \qquad i,j = 1,2,\ldots,n. \tag{14}$$

This system of differential equations is defined for all wealth distributions $c = (c_1, c_2, \ldots, c_n)$ giving rise to nonnegative incomes, and therefore it is defined for all nonnegative wealth distributions $c \varepsilon \mathbb{R}^n{}_+$. It describes the development of wealth distribution through time. Our task will be to study some characteristics of possible developments[5]).

3. The Analysis

3.1. Preliminaries

Let us first deal with some formal preliminaries to our analysis: let us dispose of the problems of existence, uniqueness, nonnegativity, and boundedness of the solutions of (14). For the sake of simplicity we will restrict our attention to nonnegative wealth distributions $c \varepsilon \mathbb{R}^n{}_+$ with an associated positive capital intensity.

Lemma 1

For any nonnegative initial wealth distribution $c_0 = (c_1(0), c_2(0), \ldots, c_n(0))$ with an associated positive capital intensity $k_0 = \Sigma a_i c_i(0)$, the system

conditions on incomes in the course of the working of the model. However, from an economic point of view, negative savings make not very much sense in the long run, and our model should be interpreted as a long run model.

Note that these difficulties are present in Stiglitz' analysis as well. He proposes to interpret $k < 0$ as a result of borrowing from foreign countries — a proposal not without problems, since $k < 0$ makes sense neither in the production function nor as a determinant of wages and interest, cf. [2, p. 384 n. 7].

[5]) In presence of Harrod-neutral technical change, η can be interpreted as the sum of technical progress and population growth.

(14) has a unique solution

(15) $$c(t) = C(t,c_0) = (C_1(t,c_0), C_2(t,c_0), \ldots, C_n(t,c_0))$$

with a corresponding time path of capital intensity

(16) $$k(t) = K(t,c_0) := \Sigma a_i C_i(t,c_0)$$

defined for all nonnegative points of time t. Both $C(t,c_0)$ and $K(t,c_0)$ are strictly positive for $t > 0$ and bounded from above $\triangle$[6]).

Proof

1. Recall that ψ is a continuously differentiable function. According to the existence theorem [6, p. 27], there exists a unique solution (15) of (14) which can be continued through $\mathbb{R}^n_+$. To point out that this solution can be continued through all positive points of time while remaining strictly positive, it is sufficient to show that c(t) cannot leave the positive orthant which is implied by the two following combined statements:

(17) There exists an $\delta > 0$ such that $k(t)\varepsilon(0,\delta)$ implies $\dot{k}(t) > 0$

(18) $$c_i(t) = 0 \text{ and } k(t) > 0 \text{ imply } \dot{c}_i(t) > 0.$$

By (17), the positivity of $K(t,c_0)$ is shown for $k_0 > 0$. Thus, $k(t) > 0$ can be presupposed in (18) to prove the positivity of $C_i(t,c_0)$ for $t > 0$, $c_0 \varepsilon \mathbb{R}^n_+$, and $k_0 > 0$.

1.1. Statement (17) is proved immediately. From (5), (6), and (12) we find

(19) $$\dot{k} = \Sigma a_i \dot{c}_i = \Sigma a_i s\{y(k) + (c_i - k)r(k), y(k)\} - \eta k.$$

$s_{11} > 0$ (i.e. convexity in y_i) and homogeneity of s imply

(20) $$\dot{k} \geq s(1,1) \cdot y(k) - \eta k.$$

Since $y(0) = 0$, $y'(0) = \infty$ (from (1)), and since y' is continuous, this proves (17).

1.2. Proposition (18) is proved by application of (2), (7), (9), and (12):

(21) $$\dot{c}_i = s\{w(k), y(k)\} > 0 \text{ if } c_i = 0 \text{ and } k > 0.$$

2. There remains to be proved the boundedness of K which implies the boundedness of C. For this purpose it is sufficient to show that there exists a $z > 0$ such that

(22) $$c(t)\,\varepsilon \mathbb{R}_+^n \text{ and } k(t) > z \text{ imply } \dot{k}(t) < 0.$$

Choose $z > 0$ such that $y(z) = \eta z$. Since $s(y_i, y) \leq y_i$, from (19) follows

$$\dot{k} \leq y(k) - \eta k$$

which is negative for $k > z$. $\triangle$

[6]) The symbol $\triangle$ indicates the end of definitions, theorems, proofs...

3.2. One-Class Equilibrium

In this and the following section we will concentrate on the equilibrium solutions of (14), i.e. those solutions remaining constant through time.

Definition 1

A one-class equilibrium is a wealth distribution $c \varepsilon \mathbb{R}^n_+$ with identical per capita wealth in all groups which remains constant through time. △

According to this definition, a one-class equilibrium is characterized by $c_1 = c_2 = \ldots = c_n$ and $c_i = 0$ for all i. From (6) and (14) we see that this equivalent to $c_i = k$ for all i and $\psi(k,k) = 0$. The equation

$$\psi(k,k) = s(1,1) \cdot y(k) - \eta k = 0 \tag{24}$$

has exactly two solutions: The trivial solution $k = 0$ and the solution $k = \underline{k}$ with $\underline{k}$ defined by

$$s(1,1) \cdot y(\underline{k}) = \eta \underline{k}, \underline{k} > 0.$$

We are now in a position to state the following theorem.

Theorem 1

The unique one-class equilibrium with positive capital intensity is given by $c = (\underline{k}, \underline{k}, \ldots, \underline{k})$ with $\underline{k}$ taken from (25). △

As a next step, the stability properties of the one-class equilibrium are to be investigated[7]).

Theorem 2

The one-class equilibrium is stable if and only if

$$s_1 \cdot r < \eta. \qquad \triangle$$

Proof

The Jacobian of (14) at $c = (\underline{k}, \underline{k}, \ldots, \underline{k})$ gives rise to the characteristic equation

$$\det \begin{pmatrix} a_1\psi_2 + \psi_1 - \lambda & a_2\psi_2 & & a_n\psi_2 \\ a_1\psi_2 & a_2\psi_2 + \psi_1 - \lambda & & a_n\psi_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_1\psi_2 & a_2\psi_2 & \ldots & a_n\psi_2 + \psi_1 - \lambda \end{pmatrix} \quad 0$$

This determinante is expanded by subtracting the second from the first row, the third from the second row, etc. After that, the first column is

[7]) The usual stability concepts, as given e.g. in [7], are used throughout. The trivial solution is neglected in the following. It can be shown that it is unstable.

added to the second column, the second column is added to the third column, and so on. Finally, equation (27) is reduced to

$$(\psi_1 - \lambda)^{n-1} \cdot (\psi_1 + \psi_2 - \lambda) = 0 \tag{28}$$

with the two roots $\lambda_1 = \psi_1$ and $\lambda_2 = \psi_1 + \psi_2$. Since $\psi_1 = s_1 r - \eta$ and $\psi_2 = s_1 \cdot (y' - r) + s_2 \cdot y' < 0$, all roots are negative if (26) holds. If $s_1 r - \lambda > 0$, one root is positive and the equilibrium is unstable. In [3, p. 74], it is shown by means of the theorem [6, p. 51] that $s_1 r - \eta = 0$ implies instability as well. △

The economic interpretation of Theorem 2 is straightforward. Assume the economy to be in one-class equilibrium, and change per capita wealth in group i by dc_i. This changes savings by $(s_1 r) \cdot dc_i$ and changes charges on per capita wealth due to population growth by $\eta \cdot dc_i$. The difference gives the change in per capita wealth which tends to eliminate the initial disturbance if (26) holds, but to increase it for $s_1 r - \eta > 0$.

Clearly, stability behaviour depends crucially on the size of the marginal propensity to save s_1. In our model, the marginal propensity to save is greater than the average propensity to save because saving depends on average income as well. So we might take the quotient of marginal and average saving propensities as a measure of this effect, as is done in the following corollary of Theorem 2.

Corollary 1

The one-class equilibrium is stable if and only if the ratio between average propensity to save and marginal propensity to save is greater than capital's share, i.e. if

$$\frac{s/y}{s_1} > \frac{r \cdot k}{y} \text{ at } c = (k, k, \ldots, k). \qquad \triangle$$

Proof

(29) follows immediately from (25) and (26). △

Thus, a one-class equilibrium is stable or unstable respectively, if — at this equilibrium position — the social influence on saving is very low or very high. This amounts to saying that the equilibrium is stable, if there is no great influence from changes in y on the consumption level. On the other hand, if this influence is sufficiently strong, the equilibrium is unstable.

3.3. Two-Class Equilibrium: Informal Discussion

Definition 2

A two-class equilibrium is a wealth distribution $c \in \mathbb{R}^n_+$ remaining constant through time which has the following property: There exist two real numbers α and β with $\alpha < \beta$ such that for each group $i \in \{1, 2, \ldots, n\}$

either $c_i = \alpha$ or $c_i = \beta$ holds. Furthermore it is required that there exists at least one $j \varepsilon \{1,2,\ldots, n\}$ with $c_j = \alpha$ and at least one $k \varepsilon \{1,2,\ldots, n\}$ with $c_k = \beta$. △

Definition 3

In a two-class equilibrium, α is called "workers' per capita wealth" and β is called "capitalists' per capita wealth". If $c_i = \alpha$, group i belongs to the workers' class ₩, being defined as

(30) $$\text{₩}: = \{i \mid c_i = \alpha\}.$$

If $c_i = \beta$, group i belongs to the capitalists' class ₵, being defined as

) $$\text{₵}: = \{i \mid c_i = \beta\}.$$

Workers' proportion in total population is

) $$\sigma: = \sum_{i \varepsilon \text{₩}} a_i, \; 0 < \sigma < 1.$$

The proportion of total capital owned by workers is

$$\tau: = \frac{\sigma \cdot \alpha}{k}$$ △

These definitions seem to be rather natural ones. In this section we will study the questions of existence and stability of two-class equilibria as defined above. It will turn out that these questions are slightly more involved in the case of a two-class equilibrium as they were in the case of the one-class equilibrium studied in the last section. In particular, the proof of our two main theorems requires a good number of lemmas, which receive their relevance only with respect to our aim. With the intention to indicate this aim, we state the theorems on existence and stability without proof at the beginning of the analysis and indicate the economic ideas underlying the methods of proof. After that we give the exact proofs.

Theorem 3

A necessary condition for the existence of a two-class equilibrium is

(34) $$r(\underline{k}) > \eta, n \geq 2$$

with $\underline{k}$ defined by (25). If there exists a sufficiently small a_j for some $j \varepsilon \{1,2,\ldots, n\}$, this condition is also sufficient. △

$n \geq 2$ is an obvious condition for the existence of a two-class equilibrium. To understand the other requirement, $r(\underline{k}) > \eta$, we have to recall that in the case of an increasing marginal propensity to save, increasing inequality of wealth and income increases savings in the economy. Therefore a two-class equilibrium will settle down at a higher capital intensity as at the one-class equilibrium would prevail, i.e. at a capital intensity $k > \underline{k}$

with a corresponding rate of interest $r(k) < r(\underline{k})$. If $r(\underline{k}) \leq \eta$, this implies $r(k) < \eta$ at the two-class equilibrium. But $r < \eta$ means that it is necessary to save more than the return from capital in order to keep per capita wealth constant. Capital does not augment "by itself" as fast as the population grows. The larger the amount of per capita wealth the greater is the difference between required savings and the return from capital. This implies that in a two-class equilibrium with $r < \eta$ the capitalists should consume less than the workers although they have a higher income. This is a contradiction to our savings assumption. Therefore, (34) is a necessary condition for the existence of a two-class equilibrium.

The argument is illustrated in Figure 1. For a given capital intensity $k > \underline{k}$ the savings curve s describes per capita savings in dependence of per capita wealth c, i.e.

$$s(c) = s\{w(k) + r(k) \cdot c, y(k)\} \tag{35}$$

with slope $s' = s_1 \cdot r < r$, the product of the marginal savings propensity and the return from capital. Since $s_{11} > 0$, we have $s'' > 0$, i.e. the savings curve is strictly convex. By (9) and (35) we have $s(0) = s\{w,y\} > 0$ for positive k. The straight line through the origin has slope η. According to (12) the change in per capita wealth is given by the difference between s(c) and $\eta \cdot c$ (the difference between per capita savings and the amount of capital which is required to match population growth). It will be shown that there exists a workers' per capita wealth $\alpha < k$ with $s(\alpha) = \eta \cdot \alpha$ if $k > \underline{k}$[8]. A necessary condition for a two-class equilibrium is the existence of a $\beta > \alpha$ with $s(\beta) = \eta \cdot \beta$, i.e. a stationary value of capitalists' per capita wealth. Such a β exists if and only if the savings curve has a slope greater than η for sufficiently high values of c. Since s' tends to r with increasing c due to the fact that increasing wealth increases income and causes the marginal savings propensity to approach unity, this implies that the capital intensity k satisfies $r(k) > \eta$. The additional requirement $k > \underline{k}$ can only be satisfied if (34) is fulfilled.

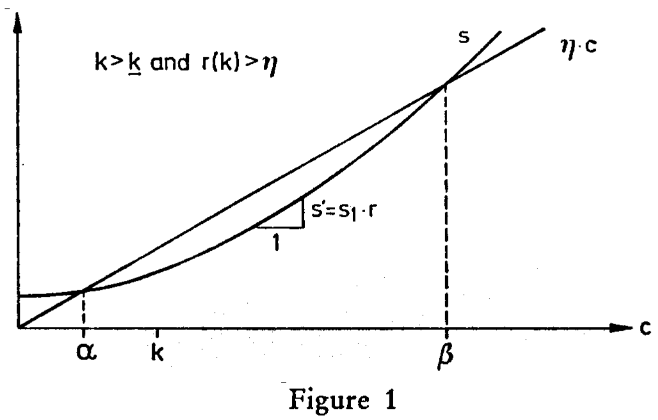

Figure 1

[8]) See Lemma 3.

Furthermore we observe that k is a weighted average of α and β with weights σ and $(1-\sigma)$, respectively (cf. (6) and (32)). Denote the capital intensity with $r = \eta$ by $\bar{k}$, i.e.

$$r(\bar{k}) = \eta \tag{36}$$

From Fig. 1 it can be seen that β tends to infinity if k tends to $\bar{k}$. Therefore, σ can be chosen arbitrarily close to unity by choosing a suitable k.

In particular $(1-\sigma) = a_j$ can be obtained if a_j is sufficiently small as postulated in Theorem 3. By choosing the classes $\mathbb{W} = \{i \neq j \mid i = 1,2,\ldots,n\}$ and $\mathbb{C} = \{j\}$, the existence of a two-class equilibrium with $(1-\sigma) = a_j$ and corresponding α, β, and k is thus established.

The following theorem deals with stability properties of two-class equilibria.

Theorem 4

1. A necessary condition for the stability of a two-class equilibrium is that the capitalists' class contains only one group.
2. All two-class equilibria satisfying the above condition with a capital intensity sufficiently close to $\bar{k}$ are stable, $\bar{k}$ being defined by (36).
3. Assume (34) to hold and a_j to be sufficiently small for one j. Then there exists a two-class equilibrium with a capitalists' class $\mathbb{C} = \{j\}$ satisfying condition 2. △

Necessity will be discussed first. Assume a two-class equilibrium with a capitalist's class containing more than one group. Since $\dot{c} = s - \eta \cdot c$ for each group, it can be seen from Fig. 1 that a slight difference of per capita wealth between two groups belonging to the capitalists' class (being a result of disturbances) is *increased* through the system. Therefore, initial disturbances cannot be eliminated. This establishes the necessity part of the theorem.

From the discussion of Theorem 3 and Figure 1 it follows that the capital intensity of a two-class equilibrium must lie betwenn $\underline{k}$ and $\bar{k}$. Now assume that there exists a two-class equilibrium with a capital intensity close to $\bar{k}$ and a capitalists' class containing only one group, say $\{j\}$. If capitalists' per capita wealth is slightly disturbed, e.g. increased, this has two effects. First, this initial disturbance will be augmented since the s-curve cuts the $\eta \cdot$ c-curve at β from below (Fig. 1). This effect will be rather small if k is sufficiently close to $\bar{k}$. The second effect is that the increase of capitalists wealth increases the aggregate capital intensity thereby shifting α to the right, since increased wages allow for higher savings of workers.

In this new situation, workers are at the left of a new α, and their wealth increases. This causes the average capital intensity to grow, the rate of interest declines, and β shifts to the right.

Since the first effect on capitalists' wealth is rather small, β will grow faster than capitalists' wealth up to a point where β is greater than capitalists' wealth. Now capitalists' wealth decreases and workers' wealth approaches a new α, say $\bar{\alpha}$. If workers' wealth is sufficiently close to its stationary value $\bar{\alpha}$, the shrinkage of capitalists' wealth will gain impact on the movement of the aggregate capital intensity, finally causing its decline. Now the whole process is reversed. The decreasing capital intensity decreases the wage rate and increases the rate of interest. Workers' wealth decreases and the shrinkage of capitalists' wealth slows down ... Finally — and after some oscillations — the original position will be restored.

If capitalists' initial wealth holdings are slightly below β, the argument will be similar. The stability mechanism which eliminates initial disturbances of workers' wealth holdings is obvious from Fig. 1: Initial disturbances of workers' wealth around α tend to be eliminated.

The third part of the theorem is easily established. Take some k arbitrarily close to $\bar{k}$ and draw the corresponding Fig. 1. The corresponding a_j is taken from the equation $k = (1 - a_j) \cdot \alpha + a_j \cdot \beta$. If k approaches $\bar{k}$, the corresponding a_j tends to zero. Now k can be chosen such that it satisfies part 2 of Theorem 4 and the corresponding a_j satisfies the condition of Theorem 3.

3.4. Two-Class Equilibrium: Formal Discussion

From (6) and Definitions 2 and 3 we derive the following lemma.

Lemma 2

In a two-class equilibrium, the following conditions are satisfied.

(37) $\psi(\alpha,k) = 0, \psi(\beta,k) = 0,\ k = \sigma \cdot \alpha + (1 - \sigma) \cdot \beta, \alpha < \beta, 0 < \sigma < 1.$ $\triangle$

Furhermore we prove

Lemma 3

There exists a solution $(\alpha(k),\beta(k),\sigma(k))$ of (37) if and only if

(38) $$k \varepsilon \mathbb{K} := \{k \mid \underline{k} < k < \bar{k}\}$$

with $\underline{k}$ defined by (25) and $\bar{k}$ defined by (36)

For $k \varepsilon \mathbb{K}$, $\alpha(k)$, $\beta(k)$, and $\sigma(k)$ are uniquely defined, positive, and differentiable. $\triangle$

Proof

1. Necessity. We prove that $k \notin \mathbb{K}$ implies condition (37) to be violated.

1.1. If $k \geq \bar{k}$, we have $r(k) \leq \eta$. Since $s_1 < 1$, $\psi_1 = s_1 \cdot r - \eta$ is negative and the equation $\psi(c,k) = 0$ cannot have two roots for c.

1.2. Assume $k \leq \underline{k}$ and (37) to hold simultaneously. Since $\psi_{11} > 0$, from (37) follows

$$\psi(k,k) < \sigma \cdot \psi(\alpha,k) + (1 - \sigma) \cdot \psi(\beta,k) = 0.$$

On the other hand, from (1) and (25) we have

$$\psi(k,k) < 0 \text{ if and only if } k > \underline{k}$$

which contradicts (39).

2. Sufficiency and uniqueness. If $k \varepsilon \mathbb{K}$, $\psi(k,k) < 0$ according to (40). Furthermore, $k \varepsilon \mathbb{K}$ implies $\psi(\infty,k) > 0$. Together with $\psi(0,k) > 0$ for $k > 0$, this proves the existence of (α,β) with $0 < \alpha < \beta$. Since $\psi_{11} > 0$, α and β are uniquely determined. The corresponding σ is $\sigma = (\beta - k)/(\beta - \alpha)$. Since ψ is a continuously differentiable function, α, β, and σ are differentiable. △

Next we prove

Lemma 4

Assume $\mathbb{K} \neq \emptyset$ and define

$$J(k) = \begin{pmatrix} \psi_1\{\alpha(k),k\} + \sigma(k) \cdot \psi_2\{\alpha(k), k\} & (1 - \sigma(k)) \cdot \psi_2\{\alpha(k),k\} \\ \sigma(k) \cdot \psi_2\{\beta(k),k\} & \psi_1\{\beta(k),k\} + (1 - \sigma(k))\, \psi_2\{\beta(k),k\} \end{pmatrix}$$

Then the following assertions hold:

1. $0 < \lim_{k \uparrow \bar{k}} \alpha(k) < \bar{k}$
2. $\lim_{k \uparrow \bar{k}} \beta(k) = \infty$
3. $\lim_{k \uparrow \bar{k}} \sigma(k) = 1$
4. $0 < \lim_{k \uparrow \bar{k}} \tau(k) < 1$
5. $-\infty < \lim_{k \uparrow \bar{k}} \psi_1\{\alpha(k),k\} < 0$
6. $\psi_1\{\beta(k), k\} > 0$ for all $k \varepsilon \mathbb{K}$, $\lim_{k \uparrow \bar{k}} \psi_1\{\beta(k), k\} = 0$
7. $\operatorname{tr} J(k) \leq -\delta$ for an $\delta > 0$ and all $k \varepsilon \mathbb{K}$ sufficiently close to $\bar{k}$
8. $\det J(k) \geq \delta$ for an $\delta > 0$ and all $k \varepsilon \mathbb{K}$ sufficiently close to $\bar{k}$
9. $\sigma'(k) \geq \delta$ for an $\delta > 0$ and all $k \varepsilon \mathbb{K}$ sufficiently close to $\bar{k}$. △

Proof

1. From (1), (2), (9), (13), and (40) we have $\psi(0,\bar{k}) = s\{w(\bar{k}),y(\bar{k})\} > 0$ and $\psi(\bar{k},\bar{k}) < 0$. Together with $\psi_{11} > 0$ throughout (from (9) and (13)) the proposition follows.

2. From (9), (13), (36), and (40) we have $\psi(\bar{k},\bar{k}) < 0, \psi_1(\infty,k) > 0$ for $k < \bar{k}$ and $\psi_1(\infty,\bar{k}) = 0$. Together with $\psi_{11} > 0$ this implies the proposition.

3. According to Lemma 3, α, β, and σ are strictly positive for all $k \varepsilon \mathbb{K}$. From Lemma 2 it follows that $0 < \alpha < \bar{k}$, i.e. α is bounded, and $\sigma = (\beta - k)/(\beta - \alpha) < 1$. Since $\lim_{k \uparrow \bar{k}} \beta(k) = \infty$, the proposition is proved.

4. This follows from 1., 3. and (33).

5. At $\bar{k}$ we have $r(\bar{k}) = \eta$. Therefore $\psi_1\{\alpha(\bar{k}),\bar{k}\} = [s_1\{w(\bar{k}) + \eta \cdot \alpha(\bar{k}), y(\bar{k})\} - 1] \cdot \eta$ is negative and bounded, since we know α to be bounded from 3.

6. Since $k \varepsilon \mathbb{K}$ implies $k > \underline{k}$, we see from (1) and (25) that $\psi_1(k,k) < 0$. From (37) we have $\alpha < k \lhd \beta$ and this establishes together with $\psi_{11} > 0$ the first statement. The second one follows from $r(\bar{k}) = \eta$ and $s_1(\infty,y) = 1$.

7. We write the trace in shorthand notation as

$$\operatorname{tr} J = \psi_1{}^\alpha + \sigma \cdot \psi_2{}^\alpha + \psi_1{}^\beta + (1 - \sigma) \cdot \psi_2{}^\beta.$$

Using (3), (13), and (37), this can be rewritten as

$$\operatorname{tr} J = \psi_1{}^\alpha + \psi_1{}^\beta + (y' - r) \cdot \{\sigma \cdot s_1{}^\alpha + (1 - \sigma) \cdot s_1{}^\beta\} +$$
$$+ y' \{\sigma \cdot s_2{}^\alpha + 1 - \sigma) \cdot s_2{}^\beta\} + (\alpha - \beta) \cdot \sigma \cdot (1 - \sigma) \cdot r' \cdot (s_1{}^\alpha - s_1{}^\beta)$$

where $s_1{}^\alpha$ is a shorthand notation for $s_1\{w(k) + r(k) \cdot \alpha(k), y(k)\}$, etc. If k approaches $\bar{k}$, we know from 5. and 6. that $\psi_1{}^\alpha$ becomes negative and $\psi_1\{\beta(\bar{k}),\bar{k}\}$ approaches zero. If k is sufficiently close to $\bar{k}$ this implies the term $(\psi_1{}^\alpha + \psi_1{}^\beta)$ to be negative. From (4), (9), and (32) it can be seen that the next term is negative, too.

(10) and (32) imply negativity for the following two terms. From (37) we know that $\alpha \lhd \beta$ which implies together with (9) that $s_1{}^\alpha < s_1{}^\beta$. Hence, together with (4) and (32) it is shown that the last term is negative. This proves proposition 7.

8. From (41) we find

$$\det J(k) = \psi_1{}^\alpha \psi_1{}^\beta + \sigma \cdot \psi_2{}^\alpha \psi_1{}^\beta + 1 - \sigma) \cdot \psi_1 \psi_2{}^\beta.$$

Since $\psi_1{}^\beta \downarrow 0$ and $\psi_1{}^\alpha$ remains bounded for $k \uparrow \bar{k}$, this implies

$$\lim_{k \uparrow \bar{k}} \det J = \lim_{k \uparrow \bar{k}} \{\sigma \psi_2{}^\alpha \psi_1{}^\beta + (1 - \sigma)\, \psi_1{}^\alpha \psi_2{}^\beta\}.$$

Furthermore, since σ, α, and hence $\psi_2{}^\alpha$ remain bounded for $k \uparrow \bar{k}$ this reduces to

$$\lim_{k \uparrow \bar{k}} \det J = \lim_{k \uparrow \bar{k}} (1 - \sigma) \cdot \psi_1{}^\alpha \psi_2{}^\beta.$$

From (37) we know that $(1 \quad \sigma) = (k - \alpha)/(\beta - \alpha)$. So we may write

$$\lim_{k \uparrow \bar{k}} \det J = \lim_{k \uparrow \bar{k}} \{(k - \alpha) \cdot \psi_1{}^\alpha\} \cdot \left\{\frac{\psi_2{}^\beta}{\beta - \alpha}\right\}.$$

Using the definition of $\psi_2{}^\beta$ we find

$$\frac{\psi_2{}^\beta}{\beta - \alpha} = \frac{s_1{}^\beta \cdot (y' - r)}{\beta - \alpha} + s_1{}^\beta \cdot \sigma \cdot r' + \frac{s_2{}^\beta \cdot y'}{\beta - \alpha} < s_1{}^\beta \cdot \sigma \cdot r' < 0,$$

Since both $(k - \alpha) \cdot \psi_1{}^\alpha$ and $s_1{}^\beta \cdot \sigma \cdot r'$ are bounded away from zero for $k \uparrow \bar{k}$ and $(k - \alpha) \cdot \psi_1{}^\alpha < 0$, the proposition is proved.

9. Differentiation of the system (which is derived from (37))

$$\begin{aligned}
\psi\{\alpha(k),k\} &= 0\\
\psi\{\beta(k),k\} &= 0\\
\sigma(k)\cdot\alpha(k) + (1-\sigma(k))\cdot\beta(k) &= k
\end{aligned}$$

with respect to k implies

$$\sigma'(k) = \frac{\det J}{(\alpha-\beta)\cdot\psi_1{}^{\alpha}\cdot\psi_1{}^{\beta}}\,. \tag{42}$$

Together with 5.—8. and $\alpha < \beta$, the proposition is proved. △

We are now sufficiently equipped to prove Theorems 3 and 4.

Proof of Theorem 3

1. Necessity. $n \geq 2$ is an obvious condition. (34) follows from Lemma 3.

2. Sufficiency. According to Lemma 4 part 9. There exists a $k^{+} < \bar{k}$ such that for all $k \,\varepsilon\, (k^{+}, \bar{k})$ the relation $\sigma'(k) > 0$ holds. Therefore, for every $a_j \,\varepsilon\, (1-\sigma(k^{+}), 0)$ there exists a two-class equilibrium with $\mathbb{C} = \{j\}$ $\mathbb{W} = \{i \neq j \mid i = 1, 2, \ldots, n\}$, $k = \sigma^{-1}(1-a_j)$ and corresponding $\alpha(k)$, $\beta(k)$ and $\sigma(k)$. (Because of $\sigma'(k) > 0$, the inverse σ^{-1} around $\bar{k}$ exists.) △

Proof of Theorem 4

Without loss of generality we assume the first m groups to belong to the workers' class and the last n-m groups to belong to the capitalists' class, i. e. we consider the system (14) at the point $c = (\underbrace{\alpha, \alpha, \ldots, \alpha}_{m}, \underbrace{\beta, \beta, \ldots, \beta}_{n\text{-}m})$.

At this point, the Jacobian gives rise to the characteristic equation

(43)

$$\det\begin{pmatrix}
a_1\psi_2{}^{\alpha} + \psi_1{}^{\alpha} - \lambda & a_2\psi_2{}^{\alpha} & \cdots & \cdots & a_n\psi_2{}^{\alpha}\\
a_1\psi_2{}^{\beta} & a_2\psi_2{}^{\alpha} + \psi_1{}^{\alpha} - \lambda & \cdot & & \cdot\\
\vdots & \vdots & & & \vdots\\
a_1\psi_2{}^{\alpha} & a_2\psi_2{}^{\alpha} & \cdots\; a_m\psi_2{}^{\alpha} + \psi_1{}^{\alpha} - \lambda & & a_n\psi_2{}^{\alpha}\\
a_1\psi_2{}^{\beta} & a_2\psi_2{}^{\beta} & \cdots & a_{m+1}\psi_2{}^{\beta} + \psi_1{}^{\beta} - \lambda\;\cdot & \cdot\; a_n\psi_2{}^{\beta}\\
\vdots & \vdots & & \cdot & \vdots\\
a_1\psi_2{}^{\beta} & a_2\psi_2{}^{\beta} & \cdot & \cdot & \cdot\; a_n\psi_2{}^{\beta} + \psi_1{}^{\beta} - \lambda
\end{pmatrix}$$

$$= 0$$

1. Necessity. Assume $m < n - 1$. The root $\lambda = \psi_1{}^\beta$ is a root of the characteristic equation because it makes the last two rows to be identical. By Lemma 4 part 7. This root is positive. Hence the equilibrium is unstable.

2. Sufficiency. Assume $m = n - 1$. Developing the determinant in a similar fashion as indicated in the proof of Theorem 2, one arrives at the following equivalent equation

$$(44) \qquad (\psi_1{}^\alpha - \lambda)^{n-2} \cdot \det (J(k) - \lambda \cdot I) = 0$$

where $J(k)$ is defined in (41) and I denotes the identity matrix. From (44) it can be seen that there are $n - 2$ identical roots $\lambda_i = \psi_1{}^\alpha$ which are negative if k is close to $\bar{k}$, see Lemma 4, part 5. The two additional roots are the roots of $J(k)$, which have negative real parts, if $\operatorname{tr} J < 0$ and $\det J > 0$. This holds true around $\bar{k}$, as proved in Lemma 4, parts 7 and 8.

3. Existence. In the proof of Theorem 3 it has been shown that there exists a two-class equilibrium with $₵ = \{j\}$ with a capital intensity close to $\bar{k}$ provided a_j to be sufficiently small. Together with 2 this proves the theorem. △

Remark

The solution referred to in Theorem 4 part 2 might be characterized by a rather small capitalists' class which nevertheless owns a considerable fraction of total capital as a consequence of Lemma 4 part 4[9]). △

3.5 Concluding Remarks on Stationary Solutions

Theorem 5

Any stationary solution is either a one-class equilibrium or a two-class equilibrium. △

Proof

Because of $\psi_{11} > 0$ there are at most two solutions x of the equation $\psi(x, k) = 0$.

Theorems 2, 4 and 5 give rise to the following. △

Corollary 2

Assume the size of the groups (i.e. the a_j's) to be sufficiently small and consider all possible stationary solutions of (14). Any equilibrium with the maximum capital intensity is stable. △

Proof

1. Assume that there exists a two-class equilibrium. We will show that any equilibrium with maximum capital intensity is a two-class equilibrium

[9]) Thanks to W. Vogt for his suggestion to consider this question.

with only one group forming the capitalists' class. This proves the proposition via application of Theorem 4.

From Lemmas 2 and 3 we find that a n y two-class equilibrium has a higher capital intensity than the one-class equilibrium. Furthermore, from Lemma 4, part 9 we have $\sigma' > 0$ around $\bar{k}$. Together with $\lim_{k \uparrow \bar{k}} \sigma(k) = 1$ this implies that a decreasing size of the capitalists' class allows for two class equilibria with higher capital intensity.

2. If there exists no two-class equilibrium, the one-class equilibrium described in Theorem 1 is the only stationary equilibrium. Since there would exist a two-class equilibrium if $r(\underline{k}) > \eta$ provided the a_j's to be sufficiently small (Theorem 3), we can assume $r(\underline{k}) \leq \eta$. This implies the stability condition (26) to be fulfilled. $\triangle$

3.6 Recurrent Solutions

Any nonstationary solution of (14) which does not approach a stationary solution, i.e. either a one-class equilibrium or a two-class equilibrium, approaches a r e c u r r e n t solution. A recurrent solution is defined as follows. Take any point which is passed by the trajectory in a certain point of time and choose an arbitrarily small distance $\delta > 0$. Then the trajectory will pass the chosen point closer than δ within a finite time interval. (For more details, see [3, pp. 25—6, 58—9] and [7, p. 38].)

Let us note that all recurrent solutions will exhibit different per capita wealth holdings for some wealth-income groups, or else the system would approach a one-class equilibrium.

Furthermore, from the convexity of $s(y_i, y)$ with regard to individual income we derive — in a similar fashion as (20) has been derived already —

$$\dot{k} > \dot{c}_i \text{ if } c_i = k \quad \text{and not all } c_i \text{ are equal.} \tag{45}$$

This means that a wealth income group with average or less than average per capita wealth will have less than average per capita wealth throughout the whole future.

Therefore recurrent solutions exhibit a property similar to the two-class property earlier analyzed: Some groups have always less than average wealth and others always own more than average wealth.

4. Acknowledgements

I thank W. V o g t for some helpful criticism of [2] which had some influence on the present formulation. Special thanks are due to W. O b e r h o f e r for his looking through the mathematical part and pointing out to me some flaws. Discussions with H. H o l l ä n d e r have been suggestive.

5. References

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6. Appendix: An Illustration

A numerically treatable discrete time version of the model can be obtained by replacing the original equations by the following versions:

$$y \quad \begin{cases} \gamma \cdot \{\alpha \cdot k^{\frac{\sigma - 1}{\sigma}} + (1 - \alpha)\}^{\frac{\sigma}{\sigma - 1}} & \text{for } \sigma \neq 1 \\ \gamma \cdot k^{\alpha} & \text{for } \sigma = 1 \end{cases}$$

$$r = \frac{\partial y}{\partial k}$$

$$s^i \quad \left((y_i)^a + (by)^a\right)^{1/a} - b \cdot y$$

$$c_i(t + 1) = \frac{1}{1 - \eta} \cdot \{s^i + c_i(t)\}$$

(1′) is a CES production function with an elasticity of substitutions σ, (4′) is the marginal productivity theory, (9′) is the savings function, and (11′) is the discrete time version of the accumulation equations (11).

Figures 2 and 3 illustrate the behaviour of the model[10]. The parameter values are $\sigma = .6$, $\alpha = .6$, $\gamma = 1.$, $\eta = .07$, $a = 2.3$, $b = 2.3$ (corresponding to an average saving propensity of .14 and a marginal saving propensity of .31 at one-class equilibrium).

Figure 2 is a typical phase diagram in the two-dimensional case: There are two groups comprising 95 and 5 per cent of total population, respectively. The beginning of each arrow indicates an initial wealth distribution and the arrow depicts the corresponding development of wealth distribution over time. The length of one arrow corresponds to 100 periods (years). There are two stable equilibria: a one-class equilibrium in the lower half of the diagram and a two-class equilibrium in the upper half. There is a saddlepoint above the one-class equilibrium. (The arrow starting close to the origin is on the 45-degree line.)

[10]) Programming aid by A. Krafft and W. Oberhofer is gratefully acknowledged.

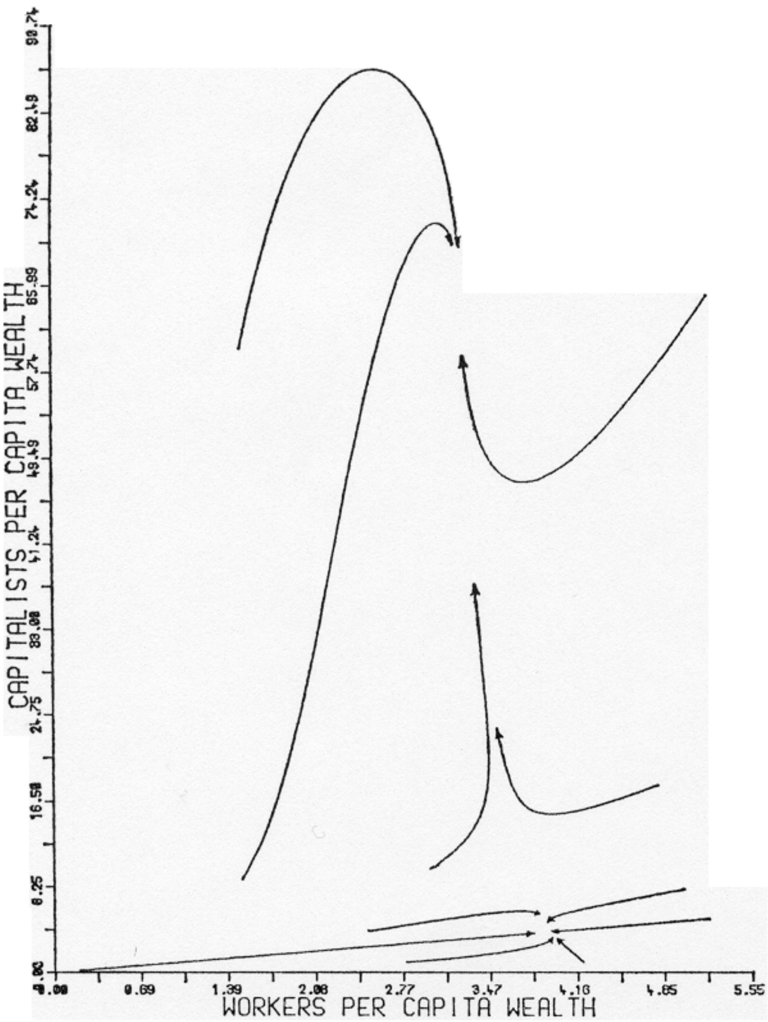

Fig. 2

Figure 3 depicts the development of wealth distribution over time in the case of eight groups. The group sizes and the initial per capita wealth holdings are the following.

Group no.	1	2	3	4	5	6	7	8
per cent of population	10	20	25	20	10	5	5	5
initial per capita wealth	.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5

The lowest curve describes the development of per capita wealth of the first group, the next one that of the second group, and so forth.

Zusammenfassung

Die lineare Sparfunktion in Stiglitz' Modell der Vermögensverteilung [1] wird durch die Annahme ersetzt, daß die durchschnittliche Sparquote jedes Wirtschaftssubjektes durch seine relative Einkommensposition bestimmt wird und daß

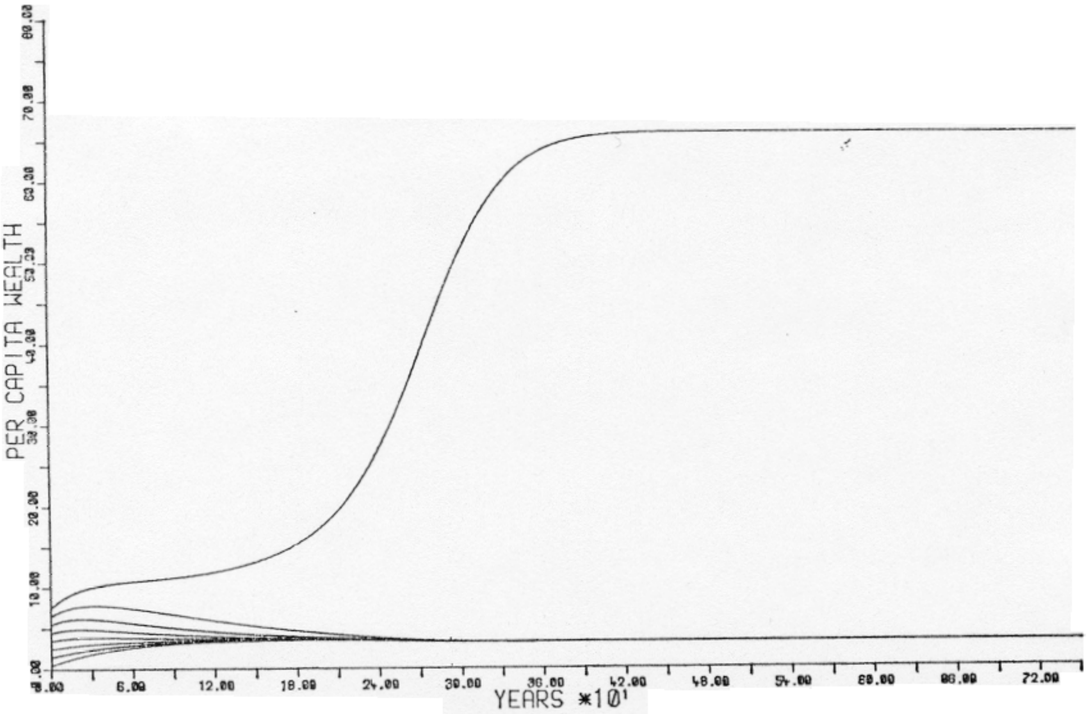

Fig. 3

die marginale Sparquote mit zunehmendem Einkommen steigt. Es wird gezeigt, daß das Modell (lokal) stabile Zweiklassenverteilungen erzeugt. Die Bedingungen dafür werden ausführlich untersucht. Das Modell erklärt also Pasinetti-Sparverhalten endogen. Eine Zwei-Klassen-Eigenschaft von nichtstationären Lösungen wird erläutert.

Summary

The linear savings function in Stiglitz' model of wealth distribution [1] is replaced by the assumption that the average savings propensity of each individual is determined by its relative income position and the marginal propensity to save is an increasing function of individual income. It is shown that the model generates locally stable two-class equilibria under certain conditions which are analyzed carefully. In other words, Pasinetti-type class savings behaviour is explained endogeneously. A two-class property of nonstationary solutions is explained.

[1] J. E. Stiglitz, Distribution of income and wealth among individuals, Econometrica 1969, pp. 382—397.

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